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Analytic Bethe ansatz related to a one-parameter family of finite-dimensional representations of the Lie superalgebra $sl(r+1|s+1)$

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Abstract. As is well known, the type 1 Lie superalgebra $sl(r+1|s+1)$ admits a one-parameter family of finite-dimensional irreducible representations. We have carried out an analytic Bethe ansatz related to this family of representations. We present formulae, which are deformations of previously proposed determinant formulae labelled by a Young superdiagram. These formulae will provide a transfer matrix eigenvalue in a dressed vacuum form related to the solutions of a graded Yang–Baxter equation, which depend not only on the spectral parameter but also on a non-additive continuous parameter. A class of transfer matrix functional relations among these formulae is briefly mentioned.

1. Introduction

The analytic Bethe ansatz [1, 2] is a powerful method that postulates the eigenvalues of transfer matrices in solvable lattice models, associated with complicated representations of underlying algebras, which are difficult to derive using other methods. We can construct them systematically in the dressed vacuum form (DVF) by using Yangians $Y(\mathcal{G})$ [3] analogue of skew-Young tableaux as in [4–6] for $\mathcal{G} = A_r,\ B_r,\ C_r$ and D_r.

Recently a similar analysis has been done [7, 8] of the Lie superalgebra $\mathcal{G} = sl(r+1|s+1)$ [9] case. These results are related to the tensor representations. A class of DVFs are obtained and shown to satisfy a set of functional relations. However, it is well known that the type 1 Lie superalgebras admit a one-parameter family of finite-dimensional irreducible representations, which is not tensor-like [10]. This is also the case with their quantum analogue. Associated with this family of representations, are solutions [11–13] of a graded Yang–Baxter equation, which depend on a non-additive continuous parameter. In [7] we pointed out a possibility of extending the DVF related to the tensor representations to the DVF related to a one-parameter family of finite-dimensional representations.

The purpose of this paper is to extend the DVF [7] to such representations. One of the simplest examples is the $sl(2|1)$ case (cf [14, 15])

$$\begin{aligned}\tilde{\mathcal{T}}^{2}_{1+c}(u) = \frac{Q_2(u-1-c)}{Q_2(u+1+c)} - \psi_3(u-1+c)\frac{Q_1(u+c)Q_2(u-1-c)}{Q_1(u+2+c)Q_2(u+1+c)} \\ -\psi_3(u-1+c)\frac{Q_1(u+4+c)Q_2(u-1-c)}{Q_1(u+2+c)Q_2(u+3+c)} \\ +\psi_3(u+1+c)\psi_3(u-1+c)\frac{Q_2(u-1-c)}{Q_2(u+3+c)}. \qquad (1.1)\end{aligned}$$

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Note that this function depends on the continuous parameter c and is still non-trivially pole free under the Bethe ansatz equation (BAE) (3.1). We shall construct a large family of the DVF with such features. The auxilliary space of the function (1.1) is related to the finite-dimensional representation with the highest weight $(1+c)(\epsilon_1+\epsilon_2)$. For $c \in \mathbb{Z}_{\geqslant 0}$, it is tensor representation labelled by the Young superdiagram with shape $((1+c)^2)$; while for $c \notin \mathbb{Z}$, it is not tensor-like.

We execute the analytic Bethe ansatz based on the BAE (3.1) associated with the distinguished simple root systems of $sl(r+1|s+1)$ [9]. Reshetikhin and Wiegmann observed [16] remarkable phenomena that the BAE can be expressed by the root system of a Lie algebra (see also [17] for the $sl(r+1|s+1)$ case). Furthermore, Kuniba *et al* [6] conjectured that the left-hand side of the BAE (3.1) can be expressed as a ratio of some 'Drinfeld polynomials' [3]. Then one can express the left-hand side of the BAE using the Kac–Dynkin label, which characterizes the quantum space. In view of the fact [10] that one can construct a finite-dimensional representation of $sl(r+1|s+1)$ whose $(r+1)$th Kac–Dynkin label takes on not only a non-negative integer value but also a *complex* value, we assume this is also the case with the left-hand side of the BAE (3.1). We introduce the Young superdiagram $\lambda \subset \mu$ [18, 19] and define the function $\mathcal{T}_{\lambda\subset\mu}(u)$ (3.19), which should be the transfer matrix in the DVF whose auxiliary space is a finite-dimensional tensor module of super Yangian [20, 21] or quantum affine superalgebra [22, 23], labelled by the skew-Young superdiagram $\lambda \subset \mu$; while the quantum space is a one-parameter family of finite-dimensional representations which is not tensor-like. One can prove the pole freeness of $\mathcal{T}^a(u) = \mathcal{T}_{(1^a)}(u)$ by the same method used in [7]. This is also the case with the function $\mathcal{T}_{\lambda\subset\mu}(u)$ since this function has determinant expressions whose matrix elements are only the functions associated with Young superdiagrams with shape $\lambda = \phi$; $\mu = (m)$ or (1^a). Correspondingly to the complex-valued $(r+1)$th Kac–Dynkin label b_{r+1}, we consider a deformation $\tilde{\mathcal{T}}_{\mu;c}(u)$ of the function $\mathcal{T}_{\mu}(u)$ by a continuous parameter c. This deformation is compatible with the so-called top-term hypothesis [5, 6]. We prove the pole freeness of the function $\tilde{\mathcal{T}}_{\mu;c}(u)$, an essential property in the analytic Bethe ansatz. Then one may think of the function $\tilde{\mathcal{T}}_{\mu;c}(u)$ as a DVF whose auxialliary space and quantum space are both parameter dependent. We present a class of transfer matrix functional relations among the DVF. It may be viewed as a kind of T-system [24] (see also [4, 6–8, 11, 15, 25–33]).

This paper is organized as follows. In section 2, we briefly review the Lie superalgebra $\mathcal{G} = sl(r+1|s+1)$. In section 3, we execute the analytic Bethe ansatz based upon the BAE (3.1) associated with distinguished simple root systems. We note that if we replace the function $\psi_a(u)$ with the one labelled by the Young superdiagram with shape (1^1), we can reproduce many of our earlier results [7] for the function $\mathcal{T}_{\lambda\subset\mu}(u)$. We prove pole freeness of the function $\tilde{\mathcal{T}}_{\mu;c}(u)$. We briefly mention functional relations for the DVF defined in this section. Our main results are relation (3.23) and theorem 3.2. Section 4 is devoted to a summary and discussion. Appendix A provides an example of the BAE for $sl(2|1)$ with the grading $p(1) = 1$, $p(2) = p(3) = 0$ and appendix B gives an example of the DVF for $sl(1|2)$.

2. The Lie superalgebra $sl(r+1|s+1)$

In this section, we briefly review the Lie superalgebra $\mathcal{G} = sl(r+1|s+1)$. A Lie superalgebra [9] is a $\mathbb{Z}_2$ graded algebra $\mathcal{G} = \mathcal{G}_{\bar{0}} \oplus \mathcal{G}_{\bar{1}}$ with a product [,], whose homogeneous elements obey the graded Jacobi identity.

There are several choices of simple root systems depending on the choices of Borel subalgebras. The simplest system of simple roots is the so-called distinguished one [9]. For example, the distinguished simple root system $\{\alpha_1, \ldots, \alpha_{r+s+1}\}$ of $sl(r+1|s+1)$ has the following form

$$\begin{aligned} &\alpha_i = \epsilon_i - \epsilon_{i+1} \qquad i = 1, 2, \ldots, r \\ &\alpha_{r+1} = \epsilon_{r+1} - \delta_1 \\ &\alpha_{j+r+1} = \delta_j - \delta_{j+1} \qquad j = 1, 2, \ldots, s \end{aligned} \tag{2.1}$$

where $\epsilon_1, \ldots, \epsilon_{r+1}; \delta_1, \ldots, \delta_{s+1}$ are the basis of the dual space of the Cartan subalgebra with the bilinear form $(\,|\,)$ such that

$$(\epsilon_i|\epsilon_j) = \delta_{i\,j} \qquad (\epsilon_i|\delta_j) = (\delta_i|\epsilon_j) = 0 \qquad (\delta_i|\delta_j) = -\delta_{i\,j} \tag{2.2}$$

with an additional constraint:

$$\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{r+1} - \delta_1 - \delta_2 - \cdots - \delta_{s+1} = 0. \tag{2.3}$$

$\{\alpha_i\}_{i \neq r+s+1}$ are even roots and α_{r+s+1} is an odd root with $(\alpha_{r+s+1}|\alpha_{r+s+1}) = 0$.

Any weight can be expressed in the following form

$$\Lambda = \sum_{i=1}^{r+1} \Lambda_i \epsilon_i + \sum_{j=1}^{s+1} \bar{\Lambda}_j \delta_j \qquad \Lambda_i, \bar{\Lambda}_j \in \mathbb{C}. \tag{2.4}$$

Let $\lambda \subset \mu$ be a skew-Young superdiagram labelled by the sequences of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ such that $\mu_i \geqslant \lambda_i : i = 1, 2, \ldots; \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$; $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant 0$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ be the conjugate of λ. There are two kinds of irreducible tensor representations for $sl(r+1|s+1)$. One of them is characterized by the Young superdiagram μ:

$$\begin{aligned} &\Lambda_i = \mu_i \qquad \text{for } 1 \leqslant i \leqslant r+1 \\ &\bar{\Lambda}_j = \eta_j \qquad \text{for } 1 \leqslant j \leqslant s+1 \end{aligned} \tag{2.5}$$

where $\eta_j = \max\{\mu'_j - r - 1, 0\}$; $\mu_{r+2} \leqslant s+1$. In this case, the Kac–Dynkin label of Λ is given [34] as follows

$$\begin{aligned} &b_j = \mu_j - \mu_{j+1} \qquad \text{for } 1 \leqslant j \leqslant r \\ &b_{r+1} = \mu_{r+1} + \eta_1 \\ &b_{j+r+1} = \eta_j - \eta_{j+1} \qquad \text{for } 1 \leqslant j \leqslant s. \end{aligned} \tag{2.6}$$

A classification theorem for the finite-dimensional irreducible unitary representations of $gl(r+1|s+1)$ was discussed in [35].

Theorem 2.1. Let Λ be a real dominant weight. The irreducible $gl(r+1|s+1)$ module $V(\Lambda)$ with the highest weight Λ is

(1) typical and type 1 unitary if

$$(\Lambda + \rho, \epsilon_{r+1} - \delta_{s+1}) > 0$$

(2) or atypical and type 1 unitary if there exists $1 \leqslant j \leqslant s+1$ such that

$$(\Lambda + \rho, \epsilon_{r+1} - \delta_j) = 0.$$

Here ρ is the graded half sum of positive roots:

$$\rho = \tfrac{1}{2} \sum_{i=1}^{r+1} (r - s - 2\mathrm{i} + 1)\epsilon_i + \tfrac{1}{2} \sum_{j=1}^{s+1} (r + s - 2j + 3)\delta_j. \tag{2.7}$$

This theorem was generalized to the type 1 quantum superalgebra $U_q(gl(r+1|s+1))$ for $q>0$ [36]. As remarked in [12], this theorem will also be valid for the type 1 quantum superalgebra $U_q(sl(r+1|s+1))$ for $q>0$. Applying theorem 2.1 to the aforementioned irreducible tensor representation, one finds that [35] Λ is typical and type 1 unitary if $\mu_{r+1}\geqslant s+1$; atypical and type 1 unitary if $\mu_{r+1}<s+1$.

There is a large class of finite-dimensional representations [10], which is not tensor-like. For example, for the aforementioned irreducible tensor representations with the highest weight Λ, a one-parameter family of irreducible representations with the highest weight (cf [12, 35])

$$\begin{aligned}&\Lambda(c)=\Lambda+c\omega\\&\omega=\epsilon_1+\epsilon_2+\cdots+\epsilon_{r+1}\end{aligned}\tag{2.8}$$

is typical and type 1 unitary if

$$(\Lambda(c)+\rho,\epsilon_{r+1}-\delta_{s+1})=\mu_{r+1}+\eta_{s+1}-s+c>0.\tag{2.9}$$

Note that the $(r+1)$th Kac–Dynkin label of $\Lambda(c)$ takes non-integer value if the parameter c is non-integral.

The dimensionality of the typical representation of $sl(r+1|s+1)$ with the highest weight Λ is given [10] as follows

$$\begin{aligned}\dim V(\Lambda)=2^{(r+1)(s+1)}&\prod_{1\leqslant i\leqslant j\leqslant r}\frac{b_i+b_{i+1}+\cdots+b_j+j-i+1}{j-i+1}\\&\times\prod_{r+2\leqslant i\leqslant j\leqslant r+s+1}\frac{b_i+b_{i+1}+\cdots+b_j+j-i+1}{j-i+1}.\end{aligned}\tag{2.10}$$

As for the atypical finite-dimensional representation, the dimensionality is smaller than the right-hand side of (2.10).

3. Analytic Bethe ansatz

Consider the following type of the BAE.

$$-\prod_{j=1}^{N}\frac{[u_k^{(a)}-w_j^{(a)}+\frac{b_j^{(a)}}{t_a}]}{[u_k^{(a)}-w_j^{(a)}-\frac{b_j^{(a)}}{t_a}]}=(-1)^{\deg(\alpha_a)}\prod_{b=1}^{r+s+1}\frac{Q_b(u_k^{(a)}+(\alpha_a|\alpha_b))}{Q_b(u_k^{(a)}-(\alpha_a|\alpha_b))}\tag{3.1}$$

$$Q_a(u)=\prod_{j=1}^{N_a}[u-u_j^{(a)}]\tag{3.2}$$

where $[u]=(q^u-q^{-u})/(q-q^{-1})$; $N_a\in\mathbb{Z}_{\geqslant 0}$; $u,w_j^{(a)}\in\mathbb{C}$; $a,k\in\mathbb{Z}$ $(1\leqslant a\leqslant r+s+1$, $1\leqslant k\leqslant N_a)$; $t_a=1$ $(1\leqslant a\leqslant r+1)$; $t_a=-1$ $(r+2\leqslant a\leqslant r+s+1)$; $b_j^{(a)}\in\mathbb{Z}_{\geqslant 0}$ $(1\leqslant a\leqslant r, r+2\leqslant a\leqslant r+s+1)$; $b_j^{(r+1)}\in\mathbb{C}$ and

$$\begin{aligned}\deg(\alpha_a)&=\begin{cases}0&\text{for even root}\\1&\text{for odd root}\end{cases}\\&=\delta_{a,r+1}.\end{aligned}\tag{3.3}$$

In this paper, we suppose that q is generic. The left-hand side of the BAE (3.1) is connected with the quantum space $W=\bigotimes_{j=1}^{N}W_j$. We assume W_j is a finite-dimensional module of super Yangian [20, 21] or quantum affine superalgebra [22, 23] whose classical counterpart is characterized by the highest weight with the Kac–Dynkin label $(b_j^{(1)}, b_j^{(2)}, \ldots, b_j^{(r+s+1)})$. We can find various kinds of BAEs, which are related to special cases of the BAE (3.1) in many literatures (for example, [11, 15, 17, 37–40]; see also [6, 14, 16, 41]). We suppose that the origin of the left-hand side of the BAE (3.1) returns to the ratio of some 'Drinfeld polynomials' $P_a(\zeta)$ $(1 \leqslant a \leqslant r+s+1)$ labelled by the Young superdiagram with shape μ:

$$P_a(\zeta) = \prod_{i=1}^{\mu_a-\mu_{a+1}} (\zeta - w + a - 2\mu_{a+1} + \mu_1 - \mu_1' - 2i + 1) \qquad 1 \leqslant a \leqslant r \tag{3.4}$$

$$P_{r+1}(\zeta) = \prod_{i=1}^{\mu_{r+1}+\eta_1} (\zeta - w + r - 2\mu_{r+1} + \mu_1 - \mu_1' + 2i) \tag{3.5}$$

$$P_{r+d+1}(\zeta) = \prod_{i=1}^{\eta_d-\eta_{d+1}} (\zeta - w - d + 2\eta_{d+1} + r + \mu_1 - \mu_1' + 2i) \qquad 1 \leqslant d \leqslant s \tag{3.6}$$

where $\mu_{r+2} \leqslant s+1$; $\prod_{i=1}^{0}(\cdots) = 1$; $w \in \mathbb{C}$. One can easily derive these polynomials (3.4)–(3.6) using the empirical procedures mentioned in [6]. Thus we obtain the following ratio of 'Drinfeld polynomial':

$$\frac{P_a\left(u + \frac{1}{t_a}\right)}{P_a\left(u - \frac{1}{t_a}\right)} = \frac{u + \frac{b_a}{t_a} - w^{(a)}}{u - \frac{b_a}{t_a} - w^{(a)}} \tag{3.7}$$

where $w^{(a)} \in \mathbb{C}$; the parameters $\{b_a\}$ denote the Kac–Dynkin label (2.6). In deriving relation (3.7), we assume the parameters $\{b_a\}$ are non-negative integers. However, as is well known [10], one can construct a finite-dimensional module whose highest weight is labelled by a Kac–Dynkin label with non-negative integers $\{b_a\}_{a\neq r+1}$ and a *complex* b_{r+1}. We then assume the parameter b_{r+1} in the relation (3.7) can take non-integer value by 'analytic continuation'. Finally by multiplying a natural q-analogue of (3.7) on each site, we obtain the left-hand side of the BAE (3.1).

We define the sets

$$\begin{aligned} &J = \{1, 2, \ldots, r+s+2\} \\ &J_+ = \{1, 2, \ldots, r+1\} \qquad J_- = \{r+2, r+3, \ldots, r+s+2\} \end{aligned} \tag{3.8}$$

with the total order

$$1 \prec 2 \prec \cdots \prec r+s+2 \tag{3.9}$$

and with the grading

$$p(a) = \begin{cases} 0 & \text{for } a \in J_+ \\ 1 & \text{for } a \in J_-. \end{cases} \tag{3.10}$$

For $a \in J$, set

$$\begin{aligned} z(a; u) &= \psi_a(u) \frac{Q_{a-1}(u+a+1) Q_a(u+a-2)}{Q_{a-1}(u+a-1) Q_a(u+a)} \qquad a \in J_+ \\ z(a; u) &= \psi_a(u) \frac{Q_{a-1}(u+2r-a+1) Q_a(u+2r-a+4)}{Q_{a-1}(u+2r-a+3) Q_a(u+2r-a+2)} \qquad a \in J_- \end{aligned} \tag{3.11}$$

where $Q_0(u) = Q_{r+s+2}(u) = 1$. Hereafter, we shall consider the case where the quantum space $W = \bigotimes_{j=1}^{N} W_j$ is a tensor-product of the module W_j labelled by the Kac–Dynkin label of the form $b_j^{(a)} = b_j \delta_{a\ r+1}$ $(1 \leqslant a \leqslant r+s+1)$. In this case, the vacuum part of the function $z(a;u)$ takes on the following form

$$\psi_a(u) = \begin{cases} 1 & \text{for } a \in J_+ \\ \displaystyle\prod_{j=1}^{N} \frac{[u - w_j + r + 1 - b_j]}{[u - w_j + r + 1 + b_j]} & \text{for } a \in J_-. \end{cases} \tag{3.12}$$

The generalization to the case of the more general quantum space will be achieved by suitable redefinition of the function $\psi_a(u)$, and such redefinition will not influence the subsequent argument. We note that one can recover a function related to those in [17] if one sets the parameters $w_j^{(a)}$, q and $\{b_j^{(a)}\}$ in the BAE (3.1) to $0, 1$ and those in (2.6) respectively. In this paper, we often express the function $z(a;u)$ as the box $\boxed{a}_u$, whose spectral parameter u will often be abbreviated. Under the BAE (3.1), we have

$$\mathrm{Res}_{u=-d+u_k^{(d)}}(z(d;u) + z(d+1;u)) = 0 \qquad 1 \leqslant d \leqslant r \tag{3.13}$$

$$\mathrm{Res}_{u=-r-1+u_k^{(r+1)}}(z(r+1;u) - z(r+2;u)) = 0 \tag{3.14}$$

$$\mathrm{Res}_{u=-2r-2+d+u_k^{(d)}}(z(d;u) + z(d+1;u)) = 0 \qquad r+2 \leqslant d \leqslant r+s+1. \tag{3.15}$$

On the skew-Young superdiagram $\lambda \subset \mu$, we assign coordinates $(i, j) \in \mathbb{Z}^2$ such that the row index i increases as we move downwards and the column index j increases as we move from left to right and that $(1, 1)$ is on the top left corner of μ. Define an admissible tableau b on the skew-Young superdiagram $\lambda \subset \mu$ as a set of elements $b(i, j) \in J$ labelled by the coordinates (i, j) mentioned above, obeying the following rule (admissibility conditions).

(1) For any elements of J_+

$$b(i, j) \prec b(i+1, j). \tag{3.16}$$

(2) For any elements of J_-

$$b(i, j) \prec b(i, j+1). \tag{3.17}$$

(3) For any elements of J

$$b(i, j) \preceq b(i, j+1) \qquad b(i, j) \preceq b(i+1, j). \tag{3.18}$$

Let $B(\lambda \subset \mu)$ be the set of admissible tableaux on $\lambda \subset \mu$. For any skew-Young superdiagram $\lambda \subset \mu$, define the function $\mathcal{T}_{\lambda\subset\mu}(u)$ as follows

$$\mathcal{T}_{\lambda\subset\mu}(u) = \sum_{b \in B(\lambda\subset\mu)} \prod_{(i,j)\in(\lambda\subset\mu)} (-1)^{p(b(i,j))} z(b(i, j); u - \mu_1 + \mu_1' - 2i + 2j) \tag{3.19}$$

where the product is taken over the coordinates (i, j) on $\lambda \subset \mu$. If we replace the vacuum part $\psi_a(u)$ (3.12) of the function $\mathcal{T}_{(1^1)}(u)$ with the one labelled by the Young superdiagram with shape (1^1), the function $\mathcal{T}_{(1^1)}(u)$ corresponds to the eigenvalue formula of the transfer matrix of the Perk–Schultz model [41–43] (see also [17]). In this case, a special case of the function $\mathcal{T}_{(1^1)}(u)$ reduces to the eigenvalue formula by the algebraic Bethe ansatz (for example, [44]: $r = 1, s = 0$ case; [45]: $r = 0, s = 1$ case; [46, 47]: $r = s = 1$ case).

The following relations should be valid [7].

$$\mathcal{T}_{\lambda\subset\mu}(u) = \det_{1\leqslant i,j\leqslant\mu_1} \left(\mathcal{T}_1^{\mu_i'-\lambda_j'-i+j}(u - \mu_1 + \mu_1' - \mu_i' - \lambda_j' + i + j - 1)\right) \tag{3.20}$$

$$= \det_{1\leqslant i,j\leqslant\mu_1'} \left(\mathcal{T}^1_{\mu_j-\lambda_i+i-j}(u - \mu_1 + \mu_1' + \mu_j + \lambda_i - i - j + 1)\right) \tag{3.21}$$

where $\mathcal{T}^a_m(u) = \mathcal{T}_{(m^a)}(u)$. These relations will be verified by the same method mentioned in [6]. We remark that the formula (3.19) reduces to the (classical) supercharacter formula if we set

$$\begin{aligned}&\boxed{a} \to \exp(\epsilon_a) \qquad \text{for } a \in J_+\\ &\boxed{a} \to \exp(\delta_{a-r-1}) \qquad \text{for } a \in J_-.\end{aligned} \tag{3.22}$$

In this case, the functions (3.20) and (3.21) reduce to the Jacobi–Trudi formulae on supersymmetric Schur functions [18, 19, 48, 49].

The following theorem is essential in the analytic Bethe ansatz.

Theorem 3.1 ([7]). For any integer a, the function $\mathcal{T}^a_1(u)$ is free of poles under the condition that the BAE (3.1) is valid†.

Applying theorem 3.1 to (3.20), one can show that $\mathcal{T}_{\lambda\subset\mu}(u)$ is free of poles under the BAE (3.1).

Owing to the admissibility conditions (3.16)–(3.18), for any Young superdiagram μ ($\mu_{r+1} \geqslant s+1$, $\mu'_1 \geqslant r+1$) and non-negative integer c, only such tableau $b \in B(\mu + (c^{r+1}))$ as $b(i,j) = i$ for $1 \leqslant i \leqslant r+1, 1 \leqslant j \leqslant c$; $b(i,j) \in J_-$ for $r+2 \leqslant i \leqslant \mu'_1, 1 \leqslant j \leqslant \mu_i$ is admissible. Then the following relation is valid:

$$\begin{aligned}\mathcal{T}_{\mu+(c^{r+1})}(u) = {} & \frac{Q_{r+1}(u-c+\mu'_1-\mu_1-r-1)}{Q_{r+1}(u+c+\mu'_1-\mu_1-r-1)}\mathcal{T}_{\hat{\mu}}(u+\mu'_1+c-r-1)\\ & \times\mathcal{H}_\nu(u-\mu_1+\mu_{r+2}-c-r-1)\end{aligned} \tag{3.23}$$

where $\mu+(c^{r+1}) = (\mu_1+c, \mu_2+c, \ldots, \mu_{r+1}+c, \mu_{r+2}, \ldots, \mu_{\mu'_1})$, $\hat{\mu} = (\mu_1, \mu_2, \ldots, \mu_{r+1})$, $\nu = (\nu_1, \nu_2, \ldots, \nu_{\mu'_1-r-1}) = (\mu_{r+2}, \mu_{r+3}, \ldots, \mu_{\mu'_1})$ and $\mathcal{H}_\nu(u)$ is the function $\mathcal{T}_\nu(u)$ whose admissible tableaux $B(\nu)$ are restricted to the sets of elements of the set J_-.

As a corollary we have (see [7])

$$\begin{aligned}\mathcal{T}^{r+1}_{c+s+1}(u) &= \mathcal{T}_{((c+s+1)^{r+1})}(u)\\ &= \frac{Q_{r+1}(u-c-s-1)}{Q_{r+1}(u+c-s-1)} \times \mathcal{T}^{r+1}_{s+1}(u+c).\end{aligned} \tag{3.24}$$

In deriving relations (3.23) and (3.24), we assume $c \in \mathbb{Z}_{\geqslant 0}$. However, these relations will also be valid for $c \in \mathbb{C}$ by 'analytic continuation'. We can easily observe this fact from the right-hand side of relations (3.23) and (3.24). Denote the right-hand side of relations (3.23) and (3.24) by $\tilde{\mathcal{T}}_{\mu;c}(u)$‡ and $\tilde{\mathcal{T}}^{r+1}_{c+s+1}(u)$, respectively for arbitrary $c \in \mathbb{C}$. A crucial condition for the function $\tilde{\mathcal{T}}_{\mu;c}(u)$ to be the eigenvalue formula of a transfer matrix is given as follows.

Theorem 3.2. For any $c \in \mathbb{C}$, the function $\tilde{\mathcal{T}}_{\mu;c}(u)$ is free of poles under the condition that the BAE (3.1) is valid.

As a corollary, we have the following.

Corollary 3.3. For any $c \in \mathbb{C}$, the function $\tilde{\mathcal{T}}^{r+1}_{c+s+1}(u)$ is free of poles under the condition that the BAE (3.1) is valid.

For any $c \in \mathbb{Z}_{\geqslant 0}$, theorem 3.2 and corollary 3.3 follow from [7], while for any $c \in \mathbb{C}$, they require proofs. In proving theorem 3.2, we use the following lemmas.

† Hereafter singularities of the vacuum parts of the DVFs, which can be removed by multiplying overall scalar functions are out of the question.

‡ See appendix B for an example of $\tilde{\mathcal{T}}_{\mu;c}(u)$.

Lemma 3.4. The function

$$\frac{\mathcal{T}_{\hat{\mu}}(u)}{Q_{r+1}(u-\mu_1)} \tag{3.25}$$

is free of poles under the condition that the BAE (3.1) is valid.

Proof. Owing to [7], the function $\mathcal{T}_{\hat{\mu}}(u)$ is free of poles under the BAE (3.1). We only have to show that function (3.25) is free of poles at $u = u_k^{(r+1)} + \mu_1 : k = 1, \ldots, N_{r+1}$. We shall show that $\mathcal{T}_{\hat{\mu}}(u)$ is divisible by $Q_{r+1}(u-\mu_1)$. In the set $\{z(a; u+\xi) : a \in J, \xi \in \mathbb{C}\}$, only $z(r+1; u-r+1-\mu_1)$ and $z(r+2; u-r+1-\mu_1)$ have $Q_{r+1}(u-\mu_1)$ in their numerators. Thus we only have to show that every term in $\mathcal{T}_{\hat{\mu}}(u)$ contains $z(r+1; u-r+1-\mu_1)$ or $z(r+2; u-r+1-\mu_1)$. Then all we have to do is to show that $b(r+1,1) = r+1$ or $r+2$ in (3.19) for $\lambda = \phi$, $\mu = \hat{\mu}$ since the argument of $z(b(i,j); u-\mu_1+r+1-2i+2j)$ in (3.19) becomes $u-r+1-\mu_1$ only when its coordinate is $(i,j) = (r+1,1)$. From the admissibility conditions, we can develop the following argument. If $b(r+1,1) \preceq r$ then $b(1,1) \prec 1$ since $b(r+1,1) \in J_+$; $b(1,1) \prec b(2,1) \prec \cdots \prec b(r+1,1) \preceq r$. This contradicts the fact that $b(1,1) \in J$. If $b(r+1,1) \succeq r+3$ then $b(r+1,\mu_{r+1}) \succ r+s+2$ since $b(r+1,1) \in J_-$; $r+3 \preceq b(r+1,1) \prec b(r+1,2) \prec \cdots \prec b(r+1,\mu_{r+1})$; $\mu_{r+1} \geqslant s+1$. This contradicts the fact that $b(r+1,\mu_{r+1}) \in J$. Thus $b(r+1,1)$ must be $r+1$ or $r+2$. In [7], we did not use the factor $Q_{r+1}(u-\mu_1)$ to prove the fact that $\mathcal{T}_{\hat{\mu}}(u)$ does not have a colour $r+1$ pole under the BAE (3.1). So division by $Q_{r+1}(u-\mu_1)$ does not influence the proof of the pole freeness of $\mathcal{T}_{\hat{\mu}}(u)$ under the BAE (3.1). Therefore function (3.25) is free of poles under the BAE (3.1). □

Lemma 3.5.

(1) The function $\mathcal{H}_\nu(u)$ is free of colour b ($b \in J - \{r+1, r+s+2\}$) poles under the condition that the BAE (3.1) is valid.

(2) The function

$$Q_{r+1}(u-\nu_1+\nu_1'+r+1)\mathcal{H}_\nu(u) \tag{3.26}$$

is free of poles under the condition that the BAE (3.1) is valid.

Proof.

(1) One can verify the following relation in the same way as relation (3.20).

$$\mathcal{H}_\nu(u) = \det_{1\leqslant i,j\leqslant \nu_1}\left(\mathcal{H}_1^{\nu_i'-i+j}(u-\nu_1+\nu_1'-\nu_i'+i+j-1)\right) \tag{3.27}$$

where $\mathcal{H}_m^a(u) = \mathcal{H}_{(m^a)}(u)$. Then we only have to show that the function $\mathcal{H}_1^a(u)$ is free of colour b ($b \in J - \{r+1, r+s+2\}$) poles under the BAE (3.1). For simplicity, we assume that the vacuum parts are formally trivial, that is, the left-hand side of the BAE (3.1) is constantly -1. The function $z(d; u) = \boxed{d}_u$ with $d \in J$ has the colour b pole only for $d = b$ or $b+1$, so we shall trace only $\boxed{b}$ or $\boxed{b+1}$ ($b \in J_- - \{r+s+2\}$). Denote S_k the partial sum of $\mathcal{H}_1^a(u)$, which contains k boxes among $\boxed{b}$ or $\boxed{b+1}$. Apparently, S_0 does not have colour b pole. Owing to relation (3.15), S_1 does not have a colour b pole ($b \neq r+1$) under the BAE (3.1).

The case $(k \geqslant 2)$: S_k is the summation of the tableaux of the form

$$
f(k,n,\xi,\zeta,u) := \begin{array}{|c|l}
\hline
\xi & \\
\hline
b & {\scriptstyle v} \\
\hline
\vdots & \\
\hline
b & {\scriptstyle v-2n+2} \\
\hline
b+1 & {\scriptstyle v-2n} \\
\hline
\vdots & \\
\hline
b+1 & {\scriptstyle v-2k+2} \\
\hline
\zeta & \\
\hline
\end{array}
= \frac{Q_{b-1}(v+2r+3-b-2n)Q_b(v+2r+4-b)}{Q_{b-1}(v+2r-b+3)Q_b(v+2r+4-b-2n)}
$$

$$
\times \frac{Q_b(v+2r+2-b-2k)Q_{b+1}(v+2r+3-b-2n)}{Q_b(v+2r+2-b-2n)Q_{b+1}(v+2r+3-b-2k)} X \qquad 0 \leqslant n \leqslant k \tag{3.28}
$$

where $\boxed{\xi}$ and $\boxed{\zeta}$ are columns with total length $a-k$, which do not contain $\boxed{b}$ and $\boxed{b+1}$; $b \in J_- - \{r+s+2\}$; $v = u+h$: h is some shift parameter and is independent of n; the function X does not have a colour b pole and is independent of n. $f(k,n,\xi,\zeta,u)$ has colour b poles at $u = -h-2r-2+b+2n+u_p^{(b)}$ and $u = -h-2r-4+b+2n+u_p^{(b)}$ for $1 \leqslant n \leqslant k-1$; at $u = -h-2r-2+b+u_p^{(b)}$ for $n=0$; at $u = -h-2r-4+b+2k+u_p^{(b)}$ for $n=k$. Obviously, colour b residue at $u = -h-2r-2+b+2n+u_p^{(b)}$ in $f(k,n,\xi,\zeta,u)$ and $f(k,n+1,\xi,\zeta,u)$ cancel each other under the BAE (3.1). Thus, under the BAE (3.1), $\sum_{n=0}^{k} f(k,n,\xi,\zeta,u)$ is free of colour b poles $(b \neq r+1)$, so is S_k.

(2) Among the boxes $\{\boxed{a} : a \in J_-\}$, only the box $\boxed{r+2}$ has colour $r+1$ pole. We shall show that the colour $r+1$ poles in $\mathcal{H}_\nu(u)$, which originate from the box $\boxed{r+2}$ are cancelled by $Q_{r+1}(u-\nu_1+\nu_1'+r+1)$. Owing to the admissibility conditions, $\boxed{r+2}$ appears consecutively only at the points $(1,1),(2,1),\ldots,(k,1) : k \leqslant \nu_1'$ in each term of $\mathcal{H}_\nu(u)$. Then the contribution of $\boxed{r+2}$ to the term of $\mathcal{H}_\nu(u)$ which contains k $\boxed{r+2}$ is

$$
\begin{aligned}
&\prod_{j=1}^{k} z(r+2, u-\nu_1+\nu_1'-2j+2) \\
&\qquad = \frac{Q_{r+1}(u-\nu_1+\nu_1'+r+1-2k)Q_{r+2}(u-\nu_1+\nu_1'+r+2)}{Q_{r+1}(u-\nu_1+\nu_1'+r+1)Q_{r+2}(u-\nu_1+\nu_1'+r+2-2k)}.
\end{aligned} \tag{3.29}
$$

Thus, the colour $r+1$ poles in $\mathcal{H}_\nu(u)$, which originated from $\boxed{r+2}$ are cancelled by $Q_{r+1}(u-\nu_1+\nu_1'+r+1)$. □

The dress part of the function $\tilde{\mathcal{T}}_{\mu;c}(u)$ carries $sl(r+1|s+1)$ weight $\Lambda(c)$ (2.8). One can observe this fact from the 'top term' [5, 6] of the function. The 'top term' is considered to be related with the highest weight vector. We speculate the 'top term' of the function $\tilde{\mathcal{T}}_{\mu;c}(u)$ for large $|q|$ is proportional to

$$
\begin{aligned}
&\frac{Q_{r+1}(u-c+\mu_1'-\mu_1-r-1)}{Q_{r+1}(u+c+\mu_1'-\mu_1-r-1)} \times \prod_{i=1}^{r+1}\prod_{j=1}^{\mu_i} z(i; u-\mu_1+\mu_1'+c-2i+2j) \\
&\qquad \times \prod_{j=1}^{s+1}\prod_{i=1}^{\eta_j} z(r+j+1; u-\mu_1+\mu_1'-c-2r-2-2i+2j) \\
&\approx q^{-2(\Lambda(c)|\sum_{a=1}^{r+s+1} N_a\alpha_a)} = q^{-2\sum_{i=1}^{r+s+1} N_i t_i b_i - 2N_{r+1}t_{r+1}c}
\end{aligned} \tag{3.30}
$$

where we omit the vacuum part. We may think of this circumstance as a generalization of the top-term hypothesis [5, 6] to the case of the non-integral highest weight. We believe that the function $\tilde{\mathcal{T}}_{\mu;c}(u)$ yields actual spectra of the transfer matrix whose auxiliary space is characterized by the highest weight $\Lambda(c)$ at least as long as the typicality condition (2.9) is satisfied. In fact, special cases of the function $\tilde{\mathcal{T}}_{\mu;c}(u)$ are in agreement with the results: for example, for $sl(2|1)$; $\mu = (2^1)$ case: [14, 15] (see also [11]). For the function $\tilde{\mathcal{T}}_{\mu;c}(u)$, one will be able to use the R matrix which is constructed by the tensor product graph method [12, 13].

As for negative integer c, care should be taken because the atypicality condition may hold. In this case, the dimensionality of the module $V(\Lambda(c))$ is no longer the one given by formula (2.10). For example, for $sl(2|2)$ case, $\tilde{\mathcal{T}}_1^2(u)$ has the form

$$\tilde{\mathcal{T}}_1^2(u) = \mathcal{T}_1^2(u) + \psi_3(u-3)\psi_3(u-1)\mathcal{T}_2^1(u). \tag{3.31}$$

In this case, the eigenvalue formula in the DVF labelled by the Young superdiagram with shape (1^2) will be the function $\mathcal{T}_1^2(u)$ rather than the function $\tilde{\mathcal{T}}_1^2(u)$.

Now we briefly mention the functional relations among the functions introduced in this section. Thanks to the Jacobi identity, the following relation holds

$$\mathcal{T}_m^a(u-1)\mathcal{T}_m^a(u+1) = \mathcal{T}_{m-1}^a(u)\mathcal{T}_{m+1}^a(u) + \mathcal{T}_m^{a-1}(u)\mathcal{T}_m^{a+1}(u) \tag{3.32}$$

where $a, m \in \mathbb{Z}_{\geqslant 0}$. This functional relation is a specialization of the Hirota bilinear difference equation [50] and it is the same as the functional relation discussed in [7] except for the vacuum part. Other functional relations in [7] are also valid except for the vacuum part. Note, however, that there are another functional relations, which arise from a one-parameter family of finite-dimensional representations. For example, $\tilde{\mathcal{T}}_{\mu;c}(u)$ satisfies

$$\tilde{\mathcal{T}}_{\mu;c}(u-d)\tilde{\mathcal{T}}_{\mu;c}(u+d) = \tilde{\mathcal{T}}_{\mu;c-d}(u)\tilde{\mathcal{T}}_{\mu;c+d}(u) \tag{3.33}$$

where $c, d \in \mathbb{C}$. For $\mu = (m^{r+1})$, $m \in \mathbb{Z}_{\geqslant s+1}$; $c = 0$; $d = 1$, this functional relation reduces to the one in [7].

4. Summary and discussion

In this paper, we have executed the analytic Bethe ansatz related to a one-parameter family of finite-dimensional representations of the type 1 Lie superalgebra $sl(r+1|s+1)$ based on the BAE (3.1) with the distinguished simple root system of $sl(r+1|s+1)$. Eigenvalue formulae of transfer matrices in DVF are proposed for a one-parameter family of finite-dimensional representations. The key is the top-term hypothesis and the observation that $(r+1)$th Kac–Dynkin label can take non-integer value. Pole freeness of the DVF was shown. Functional relations have been given for the DVF.

We emphasize that our method explained in this paper is still valid even if factors such as gauge factor, extra sign (different from $(-1)^{\deg(\alpha_a)}$ in (3.1)), etc appear in the BAE (3.1), provided that such factors do not influence the analytical property of the right-hand side of the BAE (3.1).

There is a remarkable coincidence [51, 52] between the free field realization of the generators of $U_q(\mathcal{G}^{(1)})$ associated with the classical simple Lie algebras $\mathcal{G}$ and the eigenvalue formulae [5] in the analytic Bethe ansatz. As for a Lie superalgebra $\mathcal{G}$ case, especially in relation to a one-parameter family of finite-dimensional irreducible representations, such a relation has not been discussed to date. An extensive study will be desirable.

The Lie superalgebras or their quantum analogues are not straightforward generalizations of their non-supercounterparts. They have several inequivalent sets of simple root systems depending on the choices of their Borel subalgebras. In view of this fact, we generalized [8] our result [7] to any simple root system of $sl(r+1|s+1)$. We then discussed relations among sets of the BAE for any simple root systems using the particle–hole transformation [53]. We pointed out that the particle–hole transformation is related to the reflection with respect to the element of the Weyl supergroup for odd simple root α with $(\alpha|\alpha) = 0$.

There is another type 1 superalgebra $\mathrm{osp}(2|2n)$, which also admits a one-parameter family of finite-dimensional representations (see [54, 55]). It will be an interesting problem to extend a similar analysis discussed in this paper related to $\mathrm{osp}(2|2n)$.

Functional relations among fusion transfer matrices at *finite* temperatures have been given recently in [56] and these functional relations are transformed into TBA equations (thermodynamic Bethe ansatz equations) without using string hypothesis. These TBA equations do not carry continuous parameters, which we discussed in this paper. Whether we can derive TBA equations with continuous parameters from our functional relations is an open problem.

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Appendix A. An example of the BAE for $p(1) = 1, p(2) = p(3) = 0$ grading

Based on the knowledge presented in [6], we will consider the BAE for $sl(2|1)$ with the grading $p(1) = 1$, $p(2) = p(3) = 0$. In this case, the simple roots, the sets (3.8) and the functions (3.11) have the form $\alpha_1 = \delta_1 - \epsilon_1$, $\alpha_2 = \epsilon_1 - \epsilon_2$, $J_+ = \{2, 3\}$, $J_- = \{1\}$ and

$$\boxed{1} = \psi_1(u)\frac{Q_1(u+1)}{Q_1(u-1)} \qquad \boxed{2} = \psi_2(u)\frac{Q_1(u+1)Q_2(u-2)}{Q_1(u-1)Q_2(u)} \qquad \boxed{3} = \psi_3(u)\frac{Q_2(u+2)}{Q_2(u)} \tag{A.1}$$

respectively (see [7, 8]). The top term labelled by the Young superdiagram with shape (1^1) is proportional to $\boxed{1}$, we then find that the 'Drinfeld polynomial' is

$$P_a(\xi) = \begin{cases} \xi - w & \text{for } a = 1 \\ 1 & \text{for } a = 2. \end{cases} \tag{A.2}$$

For any $b \in \mathbb{Z}_{\geqslant 1}$, the top term labelled by the Young superdiagram with shape (b^2) will be proportional to

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & 2 \\ \hline 1 & 3 & \cdots & 3 \\ \hline \end{array} = \prod_{j=1}^{b+1} \frac{Q_1(u+2j+1-b-2)}{Q_1(u+2j-1-b-2)} \tag{A.3}$$

where we omit the vacuum part. We then find that the 'Drinfeld polynomial' has the following form

$$P_a(\xi) = \begin{cases} \displaystyle\prod_{j=1}^{b+1}(\xi - w - 2j + b + 2) & \text{for } a = 1 \\ 1 & \text{for } a = 2. \end{cases} \tag{A.4}$$

Following [6], the BAE whose vacuum part corresponds to the quantum space $W = \bigotimes_{j=1}^{N} W_j$ labelled by the Young superdiagrams with shape (b^2): $j = 1$ and (1^1): $2 \leqslant j \leqslant N$ reads as follows

$$\frac{[u_k^{(1)} - w_1 - b - 1]}{[u_k^{(1)} - w_1 + b + 1]} \prod_{j=2}^{N} \frac{[u_k^{(1)} - w_j - 1]}{[u_k^{(1)} - w_j + 1]} = \frac{Q_2(u_k^{(1)} - 1)}{Q_2(u_k^{(1)} + 1)}$$

$$-1 = \frac{Q_1(u_k^{(2)} - 1)Q_2(u_k^{(2)} + 2)}{Q_1(u_k^{(2)} + 1)Q_2(u_k^{(2)} - 2)} \tag{A.5}$$

where the parameters $\{t_a\}$ are $t_1 = -1$ and $t_2 = 1$. We assume that the parameter b can take non-integer value by 'analytic continuation' as in section 3. Note that this BAE is in relation to the one in [38]. The vacuum part $\psi_a(u)$ of the function $\boxed{a}$ is determined so as to make the function $\mathcal{T}_1^1(u) = -\boxed{1} + \boxed{2} + \boxed{3}$ to be free of poles under the BAE (A.5). Up to an overall scalar function, we have

$$\psi_1(u) = 1 \tag{A.6}$$

$$\psi_2(u) = \psi_3(u) = \frac{[u - w_1 + b]}{[u - 2 - w_1 - b]} \prod_{j=2}^{N} \frac{[u - w_j]}{[u - w_j - 2]}. \tag{A.7}$$

Compare the BAE (A.5) with the one (3.1) for $sl(2|1)$ with the grading $p(1) = p(2) = 0$, $p(3) = 1$ and $b_1^{(1)} = 0$; $b_j^{(1)} = 1$: $2 \leqslant j \leqslant N$; $b_1^{(2)} = b$; $b_j^{(2)} = 0$: $2 \leqslant j \leqslant N$, whose vacuum part also originates from the quantum space $W = \bigotimes_{j=1}^{N} W_j$ labelled by the Young superdiagrams with shape (b^2): $j = 1$; (1^1): $2 \leqslant j \leqslant N$ and analytic continuation argument. Note that this BAE is also in relation to the one in [38].

Appendix B. An example of the DVF

In this section, we present an example of the DVF $\tilde{\mathcal{T}}_{\mu;c}(u)$ and theorem 3.2 for $sl(1|2)$; $\mu = (2, 1)$; $b_j = b$ (in (3.12)); $J_+ = \{1\}$; $J_- = \{2, 3\}$ case:

$$\begin{aligned}
\tilde{\mathcal{T}}_{(2,1);c}(u) &= \frac{Q_1(u-c-1)}{Q_1(u+c-1)} \mathcal{T}_{(2)}(u+c+1)\mathcal{H}_{(1)}(u-c-2) \\
&= \frac{\phi(-1-b-c+u)}{\phi(-1+b-c+u)} \left\{ -\frac{Q_1(-3-c+u)Q_2(-c+u)}{Q_1(3+c+u)Q_2(-2-c+u)} \right. \\
&\quad - \frac{\phi(1-b+c+u)\phi(3-b+c+u)Q_1(-1-c+u)Q_2(-4-c+u)}{\phi(1+b+c+u)\phi(3+b+c+u)Q_1(1+c+u)Q_2(-2-c+u)} \\
&\quad - \frac{\phi(1-b+c+u)\phi(3-b+c+u)Q_1(-3-c+u)Q_2(-c+u)}{\phi(1+b+c+u)\phi(3+b+c+u)Q_1(1+c+u)Q_2(-2-c+u)} \\
&\quad + \frac{\phi(3-b+c+u)Q_1(-1-c+u)Q_2(-4-c+u)Q_2(c+u)}{\phi(3+b+c+u)Q_1(1+c+u)Q_2(-2-c+u)Q_2(2+c+u)} \\
&\quad + \frac{\phi(3-b+c+u)Q_1(-3-c+u)Q_2(-c+u)Q_2(c+u)}{\phi(3+b+c+u)Q_1(1+c+u)Q_2(-2-c+u)Q_2(2+c+u)} \\
&\quad + \frac{\phi(3-b+c+u)Q_1(-1-c+u)Q_2(-4-c+u)Q_2(4+c+u)}{\phi(3+b+c+u)Q_1(3+c+u)Q_2(-2-c+u)Q_2(2+c+u)} \\
&\quad + \frac{\phi(3-b+c+u)Q_1(-3-c+u)Q_2(-c+u)Q_2(4+c+u)}{\phi(3+b+c+u)Q_1(3+c+u)Q_2(-2-c+u)Q_2(2+c+u)} \\
&\quad \left. - \frac{Q_1(-1-c+u)Q_2(-4-c+u)}{Q_1(3+c+u)Q_2(-2-c+u)} \right\}
\end{aligned} \tag{B.1}$$

where

$$\begin{aligned}\mathcal{T}_{(2)}(u) &= \boxed{1\quad 1} - \boxed{1\quad 2} - \boxed{1\quad 3} + \boxed{2\quad 3}\\ &= \frac{Q_1(-2+u)}{Q_1(2+u)} - \frac{\phi(2-b+u)Q_1(-2+u)Q_2(3+u)}{\phi(2+b+u)Q_1(2+u)Q_2(1+u)}\\ &\quad -\frac{\phi(2-b+u)Q_1(-2+u)Q_2(-1+u)}{\phi(2+b+u)Q_1(u)Q_2(1+u)}\\ &\quad +\frac{\phi(-b+u)\phi(2-b+u)Q_1(-2+u)}{\phi(b+u)\phi(2+b+u)Q_1(u)}\end{aligned} \tag{B.2}$$

$$\mathcal{H}_{(1)}(u) = -\boxed{2} - \boxed{3} = -\frac{\phi(1-b+u)}{\phi(1+b+u)}\left\{\frac{Q_1(-1+u)Q_2(2+u)}{Q_1(1+u)Q_2(u)} + \frac{Q_2(-2+u)}{Q_2(u)}\right\} \tag{B.3}$$

$$\phi(u) = \prod_{j=1}^{N}[u-w_j]. \tag{B.4}$$

The first term in the right-hand side of (B.1) is the top term, which is related to the highest weight $(2+c)\epsilon_1+\delta_1$. Owing to theorem 3.2, the DVF (B.1) is pole-free under the following BAE

$$\begin{aligned}&\frac{\phi(u_k^{(1)}+b)}{\phi(u_k^{(1)}-b)} = \frac{Q_2(u_k^{(1)}+1)}{Q_2(u_k^{(1)}-1)} \qquad 1\leqslant k\leqslant N_1\\ &-1 = \frac{Q_1(u_k^{(2)}+1)Q_2(u_k^{(2)}-2)}{Q_1(u_k^{(2)}-1)Q_2(u_k^{(2)}+2)} \qquad 1\leqslant k\leqslant N_2.\end{aligned} \tag{B.5}$$

We note the fact that if the parameter c is a positive integer, $\tilde{\mathcal{T}}_{(2,1);c}(u)$ has a determinant expression whose matrix elements are only the functions labelled by Young superdiagrams with one column:

$$\begin{aligned}\tilde{\mathcal{T}}_{(2,1);c}(u) &= \mathcal{T}_{(2+c,1)}(u)\\ &= \det_{1\leqslant i,j\leqslant c+2}\left(\mathcal{T}_1^{\mu_i'-i+j}(u-c-\mu_i'+i+j-1)\right)\end{aligned} \tag{B.6}$$

where $\mu_1' = 2$; $\mu_i' = 1$: $2\leqslant i\leqslant c+2$; $c\in\mathbb{Z}_{\geqslant 0}$.

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